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Overstability of a viscoelastic liquid layer with internal heat generation

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1. INTRODUCTION

THIS NOTE is concerned with thermal convection in a viscoelastic medium. The convective motion is generated by internal heat sources which give a basic temperature gradient varying with the vertical coordinate. This problem arises, in part, from the belief that thermal convection driven by internal heat sources plays an important role in the convective processes in the earth's mantle and is an important aspect of the post-accident heat removal problem that can result in the event of a core meltdown in a nuclear power reactor. Convection by internal heat sources has been studied in several papers. Closely related to the present paper are the experimental studies [1–4] and the theoretical studies [5–9]. For this problem the principle of the exchange of stabilities is considered to be valid, so the instability is manifested as a steady, cellular, convective motion, although it appears impossible at present to prove or disprove analytically the validity of this principle owing to the nonlinear temperature profile in the quiescent state due to internal heating. It is, however, expected that a layer of viscoelastic liquid can become overstable due solely to internal heating as it can do due solely to heating from below [10–12] or the dielectrophoretic forces [13] (caused by an electric field and a gradient in dielectric constant). The purpose of the present research is to evaluate the conditions under which thermally induced overstability occurs in a viscoelastic liquid with internal heat generation.

2. FORMULATION

We consider an infinite horizontal layer of a viscoelastic liquid of depth d which is heated internally by a uniform distribution of heat sources. The upper bounding surface at $z = d/2$ is perfectly conducting and maintained at a constant temperature T_1, whereas the lower bounding surface at $z = -d/2$ is thermally insulating. The liquid to be considered is assumed to have a viscoelastic nature described by the constitutive equation proposed by Oldroyd [14, 15].

Following the usual steps of linear stability theory, the equation governing small perturbations w' (vertical component of velocity) and T' (temperature) can be written as

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\nabla^2 w' - \alpha g \nabla_{\mathrm{H}}^2 T'\right) = \nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^4 w', \quad (1)$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)T' = -w'\frac{\mathrm{d}\bar{T}}{\mathrm{d}z}, \quad (2)$$

where

$$\bar{T} = T_1 + \frac{3qd^2}{8k} - \frac{q}{2k}z(z+d) \quad (3)$$

is the temperature distribution in the initial quiescent state, λ_1 is the stress relaxation time, λ_2 ($<\lambda_1$) is the strain retardation time, α is the coefficient of thermal expansion, g is the gravitational acceleration, ν is the kinematic viscosity, κ is the thermometric conductivity, q is the heat generated within the liquid per unit volume per unit time, k is the thermal conductivity and $\nabla_{\mathrm{H}}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian. Here, in addition to linearization, the usual Boussinesq approximation has been made. The associated boundary conditions are given by

$$\left.\begin{aligned} w' = \partial w'/\partial z = T' = 0 \quad &\text{at} \quad z = d/2 \\ w' = \partial w'/\partial z = \partial T'/\partial z = 0 \quad &\text{at} \quad z = -d/2. \end{aligned}\right\} \quad (4)$$

Equations (1) and (2) and the boundary conditions (4) are first rendered dimensionless by choosing d, d^2/κ, κ/d and $\nu\kappa/\alpha g d^3$ as the units of length, time, velocity and temperature respectively,

NOMENCLATURE

a	wavenumber
$\mathbf{B}(k,n), \mathbf{C}(k,n)$	particular solution vector
d	depth of the layer
D	d/dz
g	gravitational acceleration
$H_1,\ldots,H_6$	power series method constants
k	thermal conductivity
P	Prandtl number, ν/κ
q	heat generated within the liquid per unit volume per unit time
R_{I}	internal Rayleigh number, $\alpha g q d^5/k\nu\kappa$
T	temperature
w	vertical component of velocity
W	functional dependence of w on z
x, y	horizontal coordinates
z	vertical coordinate.

Greek symbols

α	coefficient of thermal expansion
Γ	elastic parameter, $\lambda_1\kappa/d^2$
θ	functional dependence of T on z
κ	thermometric conductivity
λ_1	stress relaxation time
λ_2	strain retardation time
μ	λ_2/λ_1
ν	kinematic viscosity
σ	time constant
ω	frequency.

Subscript

c	critical condition

Superscript

O	oscillatory
S	stationary
$'$	perturbed quantity
$\bar{}$	mean quantity.

and are then simplified in the usual manner by decomposing the solution in terms of normal modes, so that

$$[w', T'] = [W(z), \theta(z)] \exp [\sigma t + i(a_x x + a_y y)] \quad (5)$$

where σ is the (complex) time constant, and a_x and a_y are the (real) wavenumbers. Thus, with all variables now dimensionless, we arrive at

$$(1+\gamma\sigma)[P^{-1}\sigma(D^2-a^2)W + a^2\theta] = (1+\Gamma\mu\sigma)(D^2-a^2)^2 W, \quad (6)$$

$$(D^2-a^2-\sigma)\theta = -R_I(z+\tfrac{1}{2})W, \quad (7)$$

$$\left.\begin{array}{lll} W = DW = \theta = 0 & \text{at} & z = \tfrac{1}{2} \\ W = DW = D\theta = 0 & \text{at} & z = -\tfrac{1}{2} \end{array}\right\} \quad (8)$$

where $P = \nu/\kappa$ is the Prandtl number, $\Gamma = \lambda_1\kappa/d^2$ is an elastic parameter which may be interpreted as a Fourier number in terms of λ_1, $\mu = \lambda_2/\lambda_1$ is the ratio of the strain retardation time to the stress relaxation time, $R_I = \alpha g q d^5/k\nu\kappa$ is the internal Rayleigh number, $D = d/dz$ and $a^2 = a_x^2 + a_y^2$.

Equations (6) and (7) and the boundary conditions (8) constitute an eigenvalue system for the present problem. It is evident that when $\mu = 0$ the system reduces to that for a Maxwell liquid [11]. It is also evident that when $\Gamma = 0$ or $\mu = 1$ the system reduces to that for an ordinary viscous liquid.

3. SOLUTION

Applying the power series method, the solution of equations (6) and (7) can be expressed as

$$W = \sum_{n=1}^{6} H_n \sum_{k=1}^{\infty} \mathbf{B}(k,n) z^{k-1}, \quad (9)$$

$$\theta = \sum_{n=1}^{6} H_n \sum_{k=1}^{\infty} \mathbf{C}(k,n) z^{k-1}, \quad (10)$$

where $H_1, \ldots, H_6$ are arbitrary constants. The series coefficients $\mathbf{B}(k,n)$ and $\mathbf{C}(k,n)$ are found from equations (6) and (7) to obey the following recursion relationship:

$$\mathbf{B}(k,n) = \delta_{k,n}, \quad \text{for} \quad k \leqslant 4, \quad (11)$$

$$\begin{aligned} \mathbf{B}(k,n) = {} & \frac{1}{(k-1)(k-2)(k-3)k-4)} \\ & \times \{[(2+\Gamma\mu\sigma)a^2 + P^{-1}\sigma(1+\Gamma\sigma)] \\ & \times (k-3)(k-4)\mathbf{B}(k-2,n) - a^2[(1+\Gamma\mu\sigma)a^2 \\ & + P^{-1}\sigma(1+\Gamma\sigma)]\mathbf{B}(k-4,n) + (1+\Gamma\sigma)a^2\mathbf{C}(k-4,n)\}, \\ & \text{for} \quad k > 4, \end{aligned} \quad (12)$$

$$\mathbf{C}(k,n) = \delta_{k,n-4}, \quad \text{for} \quad k \leqslant 2, \quad (13)$$

$$\begin{aligned} \mathbf{C}(k,n) = {} & \frac{1}{(k-1)(k-2)} \Big\{(a^2+\sigma)\mathbf{C}(k-2,n) \\ & - \frac{R_I}{2}\mathbf{B}(k-2,n) - R_I\mathbf{B}(k-3,n)\Big\}, \quad \text{for} \quad k > 2, \end{aligned} \quad (14)$$

where $\delta_{i,j}$ is the Kronecker delta and $\mathbf{B}(0,n) = 0$.

Imposing the boundary conditions (3) leads to a set of six homogeneous algebraic equations for six unknown constants $H_1, \ldots, H_6$. The requirement that the determinant of the coefficients of $H_1, \ldots, H_6$ must vanish in order to ensure a non-trivial solution determines an eigenvalue equation.

4. NUMERICAL RESULTS AND DISCUSSION

If we fix the values P, Γ and μ, the eigenvalue equation gives a relation among R_I, a and σ. Since the neutral state for a stationary instability is characterized by $\sigma = 0$, the eigenvalue equation starts to give a relation between R_I and a which enables us to plot R_I against a. The lowest point of R_I as a function of a gives the critical internal Rayleigh number $R_{I_c}^S$ and the critical wavenumber a_c^S. On the other hand, since the neutral state for an oscillatory instability is characterized by $\sigma = i\omega$ with ω real, the eigenvalue equation becomes to give a relation among R_I, a, and ω. The lowest point of R_I as a function of a gives the critical internal Rayleigh number $R_{I_c}^O$, the critical wavenumber a_c^O and the corresponding critical frequency ω_c. Here the superscripts 'S' and 'O' stand for 'stationary' and 'oscillatory', respectively. The type of instability which takes place in practice will be that corresponding to the lower value of R_{I_c}. It should be noted here that as far as a stationary instability is concerned there is no distinction between an ordinary viscous liquid and a viscoelastic liquid. For a stationary instability $R_{I_c}^S$ and a_c^S have respectively the values of 2772.27 and 2.629, which are identical to the values found by Roberts [6] and Tveitereid and Palm [8].

The values of R_{I_c}, a_c and ω_c for $P = 100$ are shown in Figs. 1–3, respectively, as functions of Γ for various values of μ. Here the choice of $P = 100$ is based on the data given by Toms and Strawbridge [16] for dilute solution of polymethyl methacrylate in n-butyl acetate. It should be noted that when $P > 100$ the results are almost the same as those for $P = 100$ and the value of P is quite high for most viscoelastic liquids. The values of R_{I_c} and a_c for a stationary instability is also superimposed in Figs. 1 and 2, respectively, by a broken line. It is seen from Fig. 1 that when the elastic parameter Γ is smaller than a certain value, which depends on μ, the principle of the

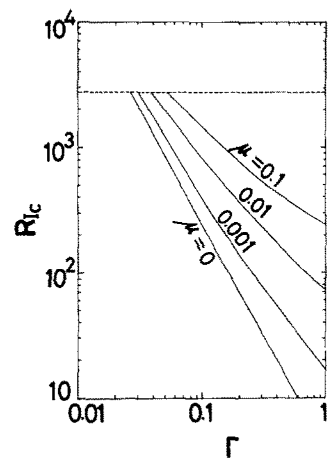

FIG. 1. The critical internal Rayleigh number R_{I_c} as a function of Γ for various values of μ when $P = 100$. —— oscillatory; ---- stationary.

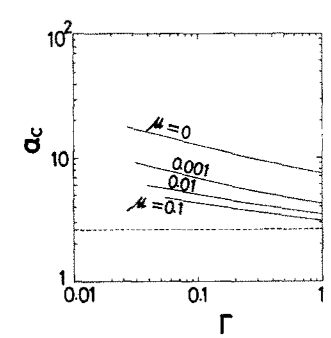

FIG. 2. The critical wavenumber a_c as a function of Γ for various values of μ when $P = 100$. —— oscillatory; ---- stationary.

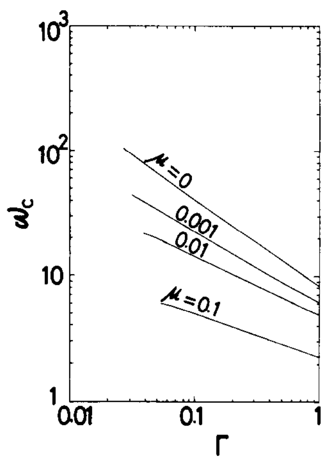

FIG. 3. The critical frequency ω_c as a function of Γ for various values of μ when $P = 100$.

exchange of stabilities does not hold, that is, instability manifests itself as overstability. In order that instability manifests itself as overstability, the value of Γ must be greater than about 0.05 for $\mu = 0.1$. This, recalling that $\Gamma = \lambda_1 \kappa/d^2$, means that the thickness d of the liquid layer must be smaller than about 0.5 mm since for most viscoelastic liquids λ_1 is at most 0.1 s [11, 16] and κ is about 0.001 cm s^{-1} [10, 11]. It therefore appears that an experimental investigation under normal laboratory conditions is not feasible. In this regard, however, it should be noted that aqueous solutions of certain recently developed polymers have relatively large relaxation times and rather low viscosities. Perhaps further development of such polymers will make oscillatory convection of more practical concern. It is also seen from Fig. 1 that the critical internal Rayleigh number R_{I_c} for the onset of overstability decreases with increase of Γ and increases with μ. Hence we may say that the elasticity of a viscoelastic liquid has a destabilizing influence on a liquid layer heated internally. It should finally be noted that the results in this note are qualitatively very similar to those in ref. [12] for the classical Bénard problem and in ref. [13] for the electrohydrodynamic instability.

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Evaluation of the importance of the relative velocity during evaporation of drops in sprays

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1. INTRODUCTION

EVAPORATION and combustion of liquid sprays in power systems invariably occurs in environments where there is a convective flow past the spray. This convective flow influences evaporation and combustion in at least two ways. First, it changes the heat and mass transfer rates between the spray as an entity, and the ambience. Secondly, it changes the geometry of the spray by entrainment of the spray periphery and recirculation of the gases surrounding the spray. These processes are all very complex and difficult to model. For this reason, guidance was sought initially from the study of individual drop evaporation and combustion. These studies [1–6] concurred with the experimental observation that a